Feynman path-integral for the damped Caldirola-Kanai action

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We propose a local Feynman path-integral representation for the damped Caldirola-Kanai action. [S1063-651X(98)12006-8]

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An interesting problem in dissipative quantum mechanics [1,2] is to find a local Feynman path integral for the classical system of a free eletron in a medium with a frictional drag proportional to velocity. In this paper we propose a formal path integral to the phenomenological Caldirola-Kanai action by following the original heuristic Feynman procedure [2] to quantize classical systems by means of a suitable sum over paths.

Let us start our analysis by considering the local Caldirola-Kanai classical action [1] of a one-dimensional free eletron of mass m moving on a medium with a frictional drag proportional to its velocity and with a positive viscosity (temperature-dependent) coefficient ν:

$$L_\nu(x(\sigma);\dot{x}(\sigma))=\int_{t'}^{t}d\sigma\exp(\nu\sigma)\left(\frac{1}{2}m\dot{x}^2(\sigma)\right). \tag{1}$$

In order to write a Feynman path-integral representation for the Feynman quantum-mechanical propagator associated to the Lagrangian Eq. (1), we follow Feynman by postulating the asymptotic Green function connecting the wave functions for infinitesimally different times $t_{k+1}-t_k=\epsilon=(t-t')/N \overset{N\to\infty}{\to} 0$

$$\psi(x_{k+1};t_{k+1})=\int_{-\infty}^{+\infty}dx_k\ \tilde{G}((x_{k+1},t_{k+1});(x_k,t_k))\psi(x_k;t_k), \tag{2}$$

where the asymptotic Green function used to define the short-time propagation is determined by the classical action, Eq. (1), with suitable prefactors:

$$\tilde{G}((x_{k+1},t_{k+1});(x_k,t_k))_{\epsilon\to 0} \sim A(t_{k+1},t_k)\exp\left\{\frac{i}{\hbar}\frac{1}{2}m\exp[\nu(at_{k+1}+bt_k)] \times\left[\frac{(x_{k+1}-x_k)^2}{\epsilon^2}\right]\epsilon\right\}. \tag{3}$$

Note that in order to analyze anomalous prefactors in the Feynman path integral for dissipative systems [2], we have introduced in the Eq. (1) the weighted rule $a+b=1$ in order to discretize the Caldirola-Kanai damping term $\exp(\nu t)$.

From Eq. (2) for $\epsilon\to 0$, we can determine the prefactor in Eq. (3) as originally done by Feynman in his heuristic description of the Feynman measure for time-independent propagators. Note that the origin of the above-mentioned ''dissipative anomaly'' is a consequence of the appearence of the discretized Caldirola-Kanai term in the expression of this purely quantum object in the Feynman path integral (propagator prefactor)

$$A(t_{k+1},t_k)=\exp\frac{\nu}{2}(at_{k+1}+bt_k)\left[\frac{m}{2\pi i\hbar(t_{k+1}-t_k)}\right]^{1/2}. \tag{4}$$

As a consequence of Eqs. (2)–(4), we can write the Green function for arbitrary different times as a Feynman path integral, as done originally in the first article of Ref. [2],

$$\begin{aligned}\tilde{G}((x,t);(x',t')) &= \lim_{N\to\infty}\int\left(\prod_{k=1}^{N-1}dx_k\right)\exp\frac{\nu}{2}\left[\sum_{k=0}^{N-1}a\left(t'+\frac{t-t'}{N}(k+1)\right)\right.\\ &\quad\left.+b\left(t'+\frac{t-t'}{N}k\right)\right]\prod_{k=0}^{N-1}\left(\frac{m}{2\pi i\hbar(t_{k+1}-t_k)}\right)^{1/2}\\ &\quad\times\exp\left\{\frac{i}{\hbar}\sum_{k=0}^{N-1}\frac{m}{2}\epsilon\exp\left[\frac{\nu}{2}(at_{k+1}+bt_k)\right]\right.\\ &\quad\left.\times\frac{(x_{k+1}-x_k)^2}{\epsilon^2}\right\}.\end{aligned} \tag{5}$$

Now we can formally define the limit in Eq. (5) as a well-defined Feynman measure over paths multiplied by a general damping anomaly factor $\exp[\nu/4(t-t')(a-b)]$, namely,

$$e^{\nu/4(t-t')(a-b)}D^F(x(\sigma)) = \lim_{N\to\infty}\left[\left(\prod_{k=1}^{N-1}dx_k\right)\left(\frac{m}{2\pi i\hbar\overline{\epsilon(}\nu,t,t')}\right)^{N/2}\right]e^{\nu/4(t-t')(a-b)}. \tag{6}$$

Note that the infinitesimal step in the factorized Feynman measure, Eq. (6), is explicitly given by the expression below and is independent of our original weighted time interval partition rule used for the dissipative term $\exp(\nu t)$ in the Caldirola-Kanai action:

$$\overline{\epsilon(}\nu,t,t')=\epsilon\exp\left[-\frac{\nu}{2}(t+t')\right]=\frac{(t-t')}{N}\exp\left[-\frac{\nu}{2}(t+t')\right]. \tag{7}$$

The above written results are a simple consequence of the following evaluations:

$$\exp\left\{\frac{\nu}{2}\left[\sum_{k=0}^{N-1}\{a[t'+\epsilon(k+1)]+b(t'+\epsilon k)\}\right]\right\}$$
$$=\exp\left\{\frac{\nu}{2}\left[\sum_{k=0}^{N-1}(at')+\sum_{k=0}^{N-1}[a\epsilon(k+1)]+\sum_{k=0}^{N-1}(bt')+\sum_{k=0}^{N-1}(b\epsilon k)\right]\right\}$$
$$=\exp\left\{\frac{\nu}{2}\left[(at')(N-1)+(bt')(N-1)+b(t-t')\frac{(N-1)}{2}+a(t-t')\frac{(N+1)}{2}\right]\right\}$$
$$=\exp\left[\frac{\nu}{2}(N-1)t'+(a-b)\frac{(t-t')}{2}+\frac{a(t-t')N}{2}+\frac{b(t-t')N}{2}\right]$$
$$=\exp\left[\frac{\nu}{4}(a-b)(t-t')\right]\exp\left[\frac{\nu}{4}(t+t')N\right]. \quad (8)$$

By substituting Eq. (8) into Eq. (5) we get our above-displayed Eq. (6).

The propagator, Eq. (5), has, thus, the dissipative anomaly found in the second article of Ref. [2] factored out by an overall anomaly factor whose exact value depends on the rule used to discretize, in Eq. (1), the term $\exp(\nu t)$ and of the initial and final time propagation. For the weighted rule it yields the result

$$\tilde{G}((x,t);(x',t'))$$
$$=e^{\nu/4(t-t')(a-b)}\int_{x(t')=x';x(t)=x}\times\mathcal{D}^F(x(\sigma))\exp\frac{i}{\hbar}\int_{t'}^{t}d\sigma e^{\nu\sigma}\left[\frac{1}{2}m\dot{x}^2(\sigma)\right]. \quad (9)$$

Note that our main result, Eqs. (6)–(9), differs somewhat from the similar one, Eq. (2.16) of the second article of Ref. [2]. Another point to stress is the similarity between the existence of a dissipative anomaly in the formal path integral Eq. (9) and the famous De-Witt anomaly in the curved space-time propagator. Let us point out the usefulness of our proposed path integral Eq. (9) with the viscosity anomaly effects factored out by calling attention to the fact that the combined Green function

$$G^{(0)}((x,t);(x',t'))=e^{-\nu/4(t-t')(a-b)}\tilde{G}[(x,t);(x',t')] \quad (10)$$

now satisfies the usual time dependent Schrödinger equation initial value problem for t and t' finite times [see Eq. (7)]

$$i\hbar\frac{\partial}{\partial t}G^{(0)}((x,t);(x',t'))=-\frac{\hbar^2}{2me^{\nu t}}\frac{d^2}{dx^2}G^{(0)}((x,t);(x',t')). \quad (11)$$

Here

$$\lim_{t\to t'}G^{(0)}((x,t);(x',t'))=\delta(x-x'). \quad (12)$$

The claim above is a general consequence of Eq. (6) defining the usual Feynman product measure [3]. At this point, we remark that by choosing the Feynman middle point rule $a=b=1/2$ in the lattice prescription for the Caldirola-Kanai path-integral propagator, we may suppress the anomaly in Eq. (9) (see Ref. [4] for similar phenomena in Feynman path integrals for curved space-time).

A simple solution of Eqs. (11),(12) is easily obtained for nonzero initial time t':

$$G^{(0)}((x,t);(x',t'))$$
$$=\left(\frac{4\pi\nu mie^{\nu t'}}{\hbar(1-e^{-\nu(t-t')})}\right)^{1/2}\exp i\frac{(x-x')^2m\nu e^{\nu t'}}{\hbar(1-e^{-\nu(t-t')})}. \quad (13)$$

The complete scheme-dependent propagator will, thus, be given by the result

$$\tilde{G}^{(0)}((x,t);(x',t'))=e^{\nu/4(t-t')(a-b)}G^{(0)}((x,t);(x',t')). \quad (14)$$

It is worth pointing out that for $a=b=1/2$, we have that the quantum probability $|\tilde{G}^{(0)}((x,t);(x',t'))|^2$ does not decay to zero at the equilibrium limit $t\to\infty$.

It is important to note that the presence of time-dependent potentials does not modify the above-displayed path-integral representation

$$\tilde{G}((x,t);(x',t'))$$
$$=e^{\nu/4(t-t')(a-b)}\int_{x(t')=x';x(t)=x}\mathcal{D}^F(x(\sigma))\exp\frac{i}{\hbar}\times\int_{t'}^{t}d\sigma e^{\nu\sigma}\{\tfrac{1}{2}m\dot{x}^2(\sigma)-V[x(\sigma),\sigma]\}. \quad (15)$$

Let us exemplify Eq. (14) by applying it to the case of the existence of a constant magnetic field perpendicular to the plane containing the particle trajectory [5]

$$\vec{A}(x,y)=-\left(\frac{1}{2}Hy\right)\vec{i}+\left(\frac{1}{2}Hx\right)\vec{j}. \quad (16)$$

Here, the particle vector position is

$$\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}. \quad (17)$$

In this two-dimensional case we have the following structure for the scheme-dependent propagator:

$$\tilde{G}((\vec{r},t);(\vec{r}\,',t'))=e^{\nu/2(t-t')(a-b)}G((\vec{r},t);(\vec{r}\,',t')) \quad (18)$$

with $G((\vec{r},t);(\vec{r}\,',t))$ now satisfying the Schrödinger time-dependent problem in view of our previous results, Eqs. (6)–(9):

$$i\hbar\frac{\partial}{\partial t}G((\vec{r},t);(\vec{r}\,',t'))$$
$$=-\frac{\hbar^2}{2me^{\nu t}}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)G((\vec{r},t);(\vec{r}\,',t'))$$
$$+\frac{\hbar eH}{2icm}e^{\nu t}\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)G((\vec{r},t);(\vec{r}\,',t')) \quad (19)$$

with

$$\lim_{t\to t'}G((\vec{r},t);(\vec{r}\,',t'))=\delta^{(2)}(\vec{r}-\vec{r}\,'). \quad (20)$$

In order to solve exactly Eqs. (19),(20), we perform the following transformation to map the above written Green function in a free Green function [6]. Namely,

$$x=[\rho(\sigma)\cos\theta(\sigma)]u+[\rho(\sigma)\sin\theta(\sigma)]v,$$
$$y=-[\rho(\sigma)\sin\theta(\sigma)]u+[\rho(\sigma)\cos\theta(\sigma)]v,$$
$$\sigma=f(t)-f(0), \quad (21)$$

with

$$\frac{d\sigma}{dt}=f'(t)=me^{\nu t}\rho^2(t),$$
$$\theta(t)=\frac{1}{2}\left(\frac{eH}{mc}\right)t,$$
$$\rho(t)=e^{-\nu t/2}\cos\left\{\sqrt{\frac{1}{4}\left[\left(\frac{eH}{mc}\right)^2-\nu^2\right]}t\right\} \quad (22)$$

and under the classical damped condition

$$\left(\frac{eH}{mc}\right)^2>\nu^2. \quad (23)$$

The Green function, Eqs. (19),(20), is, thus, given explicitly by

$$G((\vec{r},t);(\vec{r}\,',t'))$$
$$=e^{iF[u(t,x,y),v(t,x,y);\sigma(t)]}\frac{m}{2\pi i\hbar[\sigma(t)-\sigma(t')]}$$
$$\times\exp\frac{im}{2\hbar[\sigma(t)-\sigma(t')]}\{[u(t,x,y)-u(t',x',y')]^2$$
$$+[v(t,x,y)-v(t',x',y')]^2\}$$
$$\times\exp^{-iF[u(t',x',y'),v(t',x',y');\sigma(t')]}. \quad (24)$$

Note that in Eq. (24), we have used the fact that the Jacobian of the spatial coordinates, Eq. (21), is unity and the functions $u(t,x,y)$ and $v(t,x,y)$ are explicitly given by inverting Eq. (21) and (22). The complex phase function $F(u,v,\sigma)$ is explicitly given by

$$F(u,v,\sigma)=\frac{1}{2}me^{\nu t}\left\{\left[\rho(t)\sin[\theta(t)]+\frac{d}{dt}\{\rho(t)\sin[\theta(t)]\}\right.\right.$$
$$\left.+\rho(t)\cos[\theta(t)]+\frac{d}{dt}\rho(t)\cos[\theta(t)]\right][u^2(t,x,y)$$
$$\left.+v^2(t,x,y)+i\ln\rho(t)]\right\}. \quad (25)$$

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[1] Li Hua Yu and Chang-Pu Sun, Phys. Rev. A **49**, 592 (1994); K. K. Thormber and R. P. Feynman, Phys. Rev. B **1**, 4099 (1970); F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1984).
[2] R. P. Feyman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), p. 63; C. C. Gerry, J. Math. Phys. **2516**, 1820 (1984).
[3] L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981).
[4] T. Karki and A. J. Niemi, Phys. Rev. D **56**, 2080 (1997).
[5] K. K. Thormber (unpublished); X. L. Li, G. W. Ford, and R. F. O. Connell, Phys. Rev. A **41**, 5287 (1990).
[6] A. Inomata, *Remarks on the Time Transformation Techinique for Path Integration. Path Integrals from mev to Mev* (World-Scientific, Singapore, 1986), pp. 433-448; J.M.F. Bassalo and A. Nassar (private communication).
[7] A. B. Nassar *et al.*, Phys. Rev. E **56**, 1230 (1997).